J. Fluid Mech. (2002), *vol.* 459, *pp.* 207–216. © 2002 Cambridge University Press
DOI: 10.1017/S0022112002008327 Printed in the United Kingdom

Draining viscous gravity currents in a vertical fracture

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(Received 16 April 2002)

We consider the flow of instantaneous releases of a finite volume of viscous fluid in a narrow vertical fracture or Hele-Shaw cell, when there is a still narrower vertical crack in the horizontal base of the cell. The predominant motion is over the horizontal surface, but fluid also drains through the crack, progressively diminishing the volume of the current in the fracture. When the crack is shallow on the scale of the current, it saturates immediately with the draining fluid. In this case, we obtain an exact analytical solution for the motion. When the crack is deeper and does not saturate immediately, we calculate numerically the motion of the fluid in both the fracture and the crack. In each case the current advances to a finite run-out length and then retreats: we describe both phases of the motion and characterize the run-out length in terms of the controlling parameters.

1. Introduction

A common technique to increase the productivity of natural gas reservoirs involves opening a vertical fracture in the reservoir by pumping in a particle-laden slurry at high pressure: when pumping stops, the solid particles ('proppants') hold the fracture open to provide a high-permeability pathway. The dynamics of this process are rather complex, and numerical simulations must incorporate a non-Newtonian slurry rheology, infiltration through the walls of the fracture, and the settling and migration of proppants (Pearson 1994), in addition to the solid mechanics of the fracturing process. However, in recent years it has been acknowledged that there is also a role for analytical investigation of several fluid dynamical aspects of the process to complement the well-established numerical and experimental studies (Hammond 1995).

One such aspect is the spreading of injectate into a fracture which is already open, and is occupied either by some native fluid or by old slurry from which the majority of proppant particles have settled out. The injectate is generally denser than the ambient fluid and highly viscous: hence it will tend to spread along the base of the fracture, under a balance between gravitational and viscous forces. Similar viscous–gravity currents have been studied in a variety of contexts (see, for example, Huppert 1982; Didden & Maxworthy 1982; Huppert 1986; Davis & Hocking 1999, 2000).

If the fracture has a pervious rather than an impermeable base, the intruding fluid may leak slowly through it, limiting how far and how fast the current spreads. A similar phenomenon has been described by Acton, Huppert & Worster (2001), who considered currents spreading over a deep, porous horizontal surface, into which they drain. They found that when a finite volume of fluid is instantaneously released, the current initially advances, driven by the density difference between the intruding and

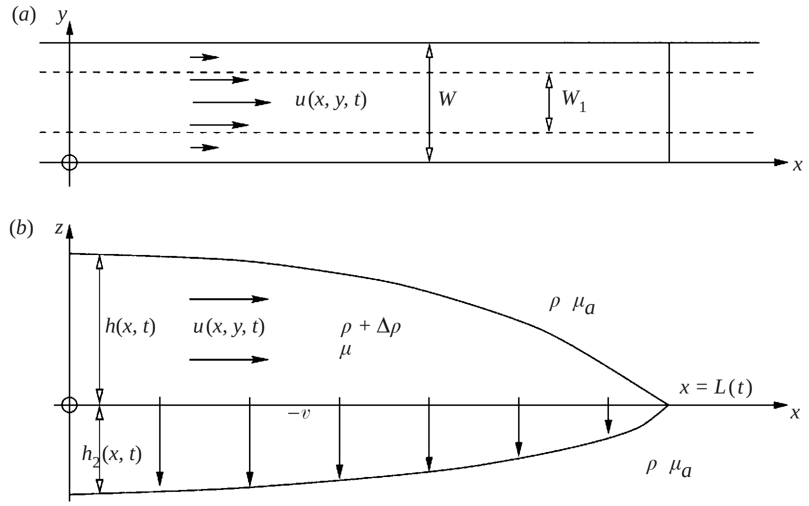

FIGURE 1. Definition diagram for a draining viscous–gravity current in a thin fracture: (*a*) plan view; (*b*) vertical cross-section. The 'fracture' corresponds to $z > 0$ and the 'crack' to $z < 0$.

ambient fluids, but that at large times drainage becomes dominant, and the current retreats before entirely draining into the underlying layer.

As a prototype for the injection of slurry into such a 'leaky' fracture, we consider the propagation of viscous fluid in a uniform vertical fracture with a thinner crack, of uniform width, in the base. The crack in turn overlies a deep region of high permeability, which is not explicitly represented in our model. We model the flow using a lubrication approximation, which yields a single nonlinear diffusion equation for the fluid depth: it is interesting to note that this equation also describes two-dimensional flow in a porous medium, and a second application of our model is to these flows, which have been considered by Huppert (1986) and subsequent studies (e.g. Woods & Mason 2000). In particular, Pritchard, Woods & Hogg (2001) investigated models of porous flow which incorporate a simplified drainage term: the current paper extends this work to regimes in which this simplified term is no longer appropriate, and the flows in both the fracture and the crack must be considered in more detail.

In § 2, we construct a model for flow in both the upper fracture and the underlying crack. We then consider two regimes. In the first regime (§ 3), the crack is sufficiently shallow that it saturates immediately as the current passes over it, while in the second (§ 4), the crack is presumed to be sufficiently deep that it never fully saturates and the evolving saturated region must be considered in more detail.

2. Derivation of the governing equations

The geometry of the flow is shown in figure 1. We consider an intrusion of density $\rho + \Delta\rho$ and viscosity μ into an ambient of density ρ and viscosity $\mu_a < \mu$. The z-coordinate is vertical and the y-coordinate is horizontal and perpendicular to the along-fracture coordinate x. The width of the fracture is W, and the width of the underlying crack is $W_1 < W$. The velocity in the fracture in the x-direction is denoted by u, and the fluid drains through the base of the fracture with a volume rate per unit area $-v$. The length of the current is denoted by $L(t)$, and the saturated region of the underlying crack is given by $-h_2(x,t) < z < 0$. Both the fracture and the region underlying the crack are presumed to be deep, so return flows may be neglected.

2.1. *Flow in the fracture*

We assume that W is much less than the characteristic depth H of the current, which is itself much less than the length of the current, $W \ll H \ll L$. The pressure can then be shown to be hydrostatic, $p = p_0 + \rho g(H_0 - h) + (\rho + \Delta\rho)g(h - z)$, where H_0 and p_0 are a reference depth and pressure respectively; and the shear of the velocity field in the y-direction is then much greater than the shear in the z-direction. The pressure in the ambient fluid, which initially extends throughout $z < 0$ as well as $z > 0$, is also hydrostatic, $p_a = p_0 + \rho g(H_0 - z)$.

Close to the nose of the current $x = L(t)$, the effects of surface tension must be included to obtain a formally accurate description of the current (Hocking 1983): this is particularly important in geometries in which the gradient of the current is infinite at the nose (such as those considered by Huppert 1982). However, the correction terms are generally small and do not affect the global dynamics of the current: we therefore neglect capillary forces throughout this study.

If the Reynolds number is much less than unity, $\rho UH/\mu \ll 1$ where U is a typical velocity in the current, the motion is governed by a balance between the streamwise pressure gradient and the lateral gradient of the viscous stresses,

$$\frac{\partial p}{\partial x} = \mu \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right), \tag{2.1}$$

with the no-slip boundary condition $u = 0$ on $y = 0$ and, by symmetry, $\partial u/\partial y = 0$ on $y = W/2$. The appropriate solution in $0 \leqslant y \leqslant W/2$ is

$$u(x, y, t) = -\frac{1}{2}\left(\frac{g\Delta\rho}{\mu}\right)\frac{\partial h}{\partial x}\left[\left(\frac{W}{2}\right)^2 - \left(\frac{W}{2} - y\right)^2\right], \tag{2.2}$$

and we substitute this into the mass conservation equation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left[\int_0^h \frac{2}{W}\left(\int_0^{W/2} u\,\mathrm{d}y\right)\mathrm{d}z\right] = v, \tag{2.3}$$

to obtain

$$\frac{\partial h}{\partial t} = \beta \frac{\partial}{\partial x}\left[h\frac{\partial h}{\partial x}\right] + v, \quad \text{where} \quad \beta = \frac{W^2}{12}\left(\frac{g\Delta\rho}{\mu}\right). \tag{2.4}$$

For an instantaneous finite release of fluid, we require that the mass flux is zero at the tail of the current, and thus $\partial h/\partial x = 0$ at $x = 0$.

2.2. *Flow in the crack: drainage term*

We consider two regimes: the first applies if the crack is sufficiently shallow that it saturates immediately with fluid as the current passes over it; and the second applies if the crack is deep, and the extent of the saturated region must be considered.

The derivation of the drainage term is slightly simpler in the first regime. We consider a crack of depth b: for purely vertical flow, conservation of mass requires that $\partial^2 p/\partial z^2 = 0$, and therefore that the pressure gradient in the crack is given by

$$\frac{\partial p}{\partial z} = g\left[\Delta\rho \frac{h}{b} - \rho\right], \tag{2.5}$$

leading to the balance of forces

$$\mu\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = -g(\rho+\Delta\rho) - \frac{\partial p}{\partial z} = g\Delta\rho\left(\frac{h}{b}+1\right), \tag{2.6}$$

where w is the vertical velocity in the crack. Integrating with respect to y we obtain

$$w(x,y,t) = \frac{1}{2}\left(\frac{g\Delta\rho}{\mu}\right)\left(\frac{h}{b}+1\right)\left[\left(\frac{W_1}{2}-y\right)^2 - \left(\frac{W_1}{2}\right)^2\right]. \tag{2.7}$$

This drainage term will provide a leading-order description of the flow in the crack as long as the vertical drainage velocity w is much greater than the horizontal velocity in the crack u_1 which is driven by the horizontal pressure gradient imposed on $z = 0$. Equation (2.2) indicates that we may expect terms associated with u_1 to scale as

$$u_1 \sim \left(\frac{g\Delta\rho}{\mu}\right) W_1^2 \left|\frac{\partial h}{\partial x}\right|, \tag{2.8}$$

and so the condition $w \gg u_1$ is automatically satisfied if $H/L \ll 1$.

We also require that in the fracture itself, the vertical velocities associated with the drainage are much smaller than the horizontal velocities driven by the hydrostatic pressure gradient, $w \ll u$. Thus for the description to be consistent, we must have $W_1^2/W^2 \ll H/L \ll 1$. If these conditions are satisifed, flow in the fracture will be horizontal to leading order and flow in the crack will be vertical to leading order.

The mass loss term v is now given by averaging across the width of the crack,

$$v = -\frac{1}{12}\left(\frac{g\Delta\rho}{\mu}\right)\frac{W_1^3}{W}\left(1+\frac{h}{b}\right). \tag{2.9}$$

We may also now quantify the regime in which this form of the drainage term is valid: we require that the timescale associated with motion in the upper layer is much greater than the timescale for drainage through the crack, and thus that $L/\beta \gg b/v$, which is satisfied if $L/b \gg W^3/W_1^3$.

In the second regime, the crack is sufficiently deep that it does not saturate instananeously with fluid (i.e. $(L/b)(W_1/W)^3 = O(1)$ or smaller), and the derivation is very similar, except that the boundary conditions on pressure are applied at $z = 0$ and at $z = -h_2(x,t)$, the lower surface of the saturated region of the crack. Thus, the pressure gradient becomes

$$\frac{\partial p}{\partial z} = g\left[\Delta\rho\frac{h}{h_2} - \rho\right], \tag{2.10}$$

and proceeding as before we obtain the drainage term

$$v = -\frac{1}{12}\left(\frac{g\Delta\rho}{\mu}\right)\frac{W_1^3}{W}\left(1+\frac{h}{h_2}\right). \tag{2.11}$$

We must also consider the evolution of the saturated region, which for purely vertical flow is given simply by the conservation of fluid mass,

$$\frac{\partial h_2}{\partial t} = -\frac{W}{W_1}v = \frac{W_1^2}{12}\left(\frac{g\Delta\rho}{\mu}\right)\left(1+\frac{h}{b}\right). \tag{2.12}$$

2.3. *The stability of the lower interface*

The interface at $z = -h_2$ may be subject to a gravitational instability if the denser and more viscous overlying fluid descends too slowly. Neglecting surface tension, the criterion for this instability to be suppressed is

$$\frac{W_1^2}{12} g\Delta\rho < (\mu - \mu_a)\frac{\partial h_2}{\partial t} \tag{2.13}$$

(Saffman & Taylor 1958). Substituting for $\partial h_2/\partial t$ using equation (2.12) yields

$$\frac{h}{h_2} > \frac{\mu_a}{\mu - \mu_a}. \tag{2.14}$$

For a highly viscous slurry propagating into a less viscous ambient, $\mu_a/\mu \ll 1$, so while $h/h_2 \gtrsim \mu_a/\mu$, the interface remains stable and the derivation of the drainage term is consistent.

2.4. *Non-dimensionalization*

For a finite instantaneous release of dense fluid, with volume $V(t)$ and initial volume $\mathscr{V}$ per unit width of the fracture, we define the dimensionless variables

$$\hat{h} = \frac{h}{\mathscr{V}^{1/2}}, \quad \hat{h}_2 = \frac{h_2}{\mathscr{V}^{1/2}}, \quad \hat{x} = \frac{x}{\mathscr{V}^{1/2}}, \quad \hat{t} = \frac{\beta}{\mathscr{V}^{1/2}}t, \quad \hat{v} = \frac{v}{\beta}, \tag{2.15}$$

to obtain the scaled governing equation

$$\frac{\partial \hat{h}}{\partial \hat{t}} = \frac{\partial}{\partial \hat{x}}\left[\hat{h}\frac{\partial \hat{h}}{\partial \hat{x}}\right] + \hat{v}. \tag{2.16}$$

The boundary conditions are given by

$$\frac{\partial \hat{h}}{\partial \hat{x}} = 0 \quad \text{at} \quad \hat{x} = 0, \quad \hat{h} = 0 \quad \text{at} \quad \hat{x} = \hat{L}(\hat{t}), \tag{2.17}$$

and the initial dimensionless volume $\hat{V}(0) = 1$.

For a shallow crack, the drainage term becomes $\hat{v} = -\lambda\hat{h} - \epsilon$, where

$$\epsilon = \frac{1}{12}\left(\frac{g\Delta\rho}{\beta\mu}\right)\frac{W_1^3}{W} = \frac{W_1^3}{W^3} \quad \text{and} \quad \lambda = \frac{\mathscr{V}^{1/2}}{b}\epsilon. \tag{2.18}$$

For a deep crack, the governing equations become

$$\frac{\partial \hat{h}}{\partial \hat{t}} = \frac{\partial}{\partial \hat{x}}\left[\hat{h}\left(\frac{\partial \hat{h}}{\partial \hat{x}}\right)\right] - \lambda_1\left(\frac{\hat{h}}{\hat{h}_2} + 1\right) \quad \text{and} \quad \frac{\partial \hat{h}_2}{\partial \hat{t}} = \lambda_2\left(\frac{\hat{h}}{\hat{h}_2} + 1\right), \tag{2.19}$$

where

$$\lambda_1 = \frac{1}{12}\left(\frac{g\Delta\rho}{\beta\mu}\right)\frac{W_1^3}{W} = \frac{W_1^3}{W^3} \quad \text{and} \quad \lambda_2 = \frac{W_1^2}{W^2}. \tag{2.20}$$

We note at this point that the system we have described is analagous to flow in a porous medium over a horizontal layer of lower permeability, while the drainage terms have the same non-dimensional form as those derived by Acton *et al.* (2001) for drainage into a porous layer. The permeabilities in the upper and lower layers correspond to $K \equiv W^2/12$ and $K_1 \equiv W_1^2/12$, while the ratio of the effective porosities of the upper and lower layers is given by $\phi/\phi_1 \equiv W/W_1$. Thus the results described here may be interpreted as an extension of the model for flow in porous media discussed by Pritchard *et al.* (2001).

2.5. *Similarity solutions for a non-draining current*

For a current which propagates over an impermeable base, and so does not lose mass by drainage, equation (2.4) admits a similarity solution (Pattle 1959),

$$\hat{h}(\hat{x},\hat{t}) = \tfrac{1}{6}\hat{t}^{-1/3}\left(9^{2/3} - \frac{\hat{x}^2}{\hat{t}^{2/3}}\right). \tag{2.21}$$

This has been confirmed experimentally (Huppert 1986; Huppert & Woods 1995; Woods & Mason 2000) to provide a good description of the flow of a viscous liquid in a Hele-Shaw cell. Pritchard *et al.* (2001) investigated numerically the convergence to this solution for a non-singular initial release, and confirmed that it is rapidly attracting for a range of initial conditions.

3. Drainage through a shallow crack

We recall that the drainage term in this case has the form $\hat{v} = -\lambda\hat{h} - \epsilon$. We can further rescale the equation, defining $h^* = \lambda^{-1/3}\hat{h}$, $t^* = \lambda\hat{t}$, $x^* = \lambda^{1/3}\hat{x}$ and $\epsilon^* = \epsilon\lambda^{-4/3} = (W/W_1)(b/\mathscr{V}^{1/2})^{4/3}$, to eliminate λ from both the equation and the initial condition on volume, and obtain

$$\frac{\partial h^*}{\partial t^*} = \frac{\partial}{\partial x^*}\left(h^*\frac{\partial h^*}{\partial x^*}\right) - h^* - \epsilon^*. \tag{3.1}$$

We recall that the regime required for this description to be valid is $L/b \gg (W/W_1)^3$, and this may be rewritten as $\epsilon^* \ll b/\mathscr{V}^{1/2}$; we are therefore concerned only with small values of ϵ^*. However, we shall see that even small values of $\epsilon^* > 0$ can make a qualitative difference to the behaviour of the current.

We approach this problem by seeking a solution of the form

$$h^*(x^*,t^*) = \exp(-t^*)\tfrac{1}{6}\tau^{*-1/3}\left(9^{2/3} - \frac{x^{*2}}{\tau^{*2/3}}\right) + \epsilon^* h_1^*(x^*,t^*), \tag{3.2}$$

where the leading-order term corresponds to the solution for pressure-driven drainage obtained by Pritchard *et al.* (2001), and where $\tau^* = 1 - e^{-t^*}$. We employ the ansatz $h_1^* = h_1^*(t^*)$, under which all spatial dependence in equation (3.1) vanishes, and we obtain an ordinary differential equation for $h_1^*(t^*)$. The initial condition is $h_1^*(0) = 0$, because as $t^* \to 0$, $h^* \to \infty$, and so the term $-h^*$ dominates the term -1 in the drainage flow v^* and the problem reduces to that of purely 'pressure-driven' drainage, described by the leading-order term.

We obtain the solution

$$h_1^*(t^*) = \frac{(e^{-t^*}-1)^{2/3}}{e^{t^*}-1}\int_0^{t^*} e^u(e^{-u}-1)^{1/3}\,du, \tag{3.3}$$

which can be expressed in closed form as

$$h_1^*(\tau) = -1 - \frac{1-\tau^*}{\tau^{*1/3}}\left[\log\left(\frac{(1-\tau^{*1/3})^{1/2}}{(1-\tau^*)^{1/6}}\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1+2\tau^{*1/3}}{\sqrt{3}}\right) + \frac{\pi}{6\sqrt{3}}\right]. \tag{3.4}$$

We note that in the regime $(1-\tau^*) \ll 0$, $h_1^*(t^*) \sim -1 + O((1-\tau^*)\log(1-\tau^*))$, and thus $h_1^*(t^*) \to -1$ as $t^* \to \infty$, so the perturbation term is smaller than ϵ^* for all time.

An interesting feature of this solution is that, as for the solutions presented by Pritchard *et al.* (2001), the parabolic shape of the current is preserved: in fact,

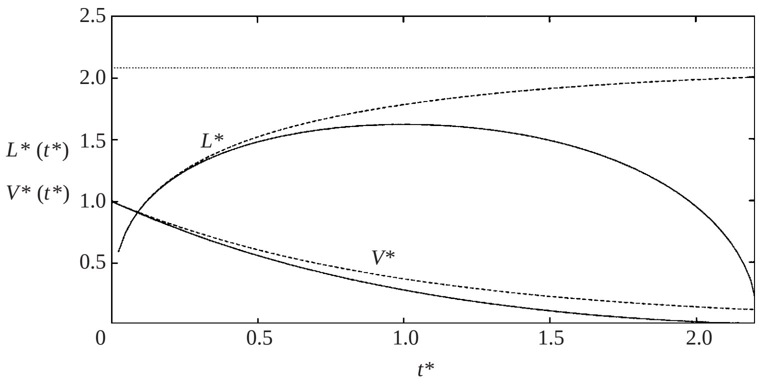

FIGURE 2. The length $L^*(t^*)$ and volume $V^*(t^*)$ of draining currents with $\epsilon^* = 0.1$ (—) and $\epsilon^* = 0$ (- -). $L^*(t^*)$ is initially increasing, and $V^*(0) = 1$ for both cases. The run-out length for the case $\epsilon = 0$, $L^*_\infty = 9^{1/3}$, is also shown (···).

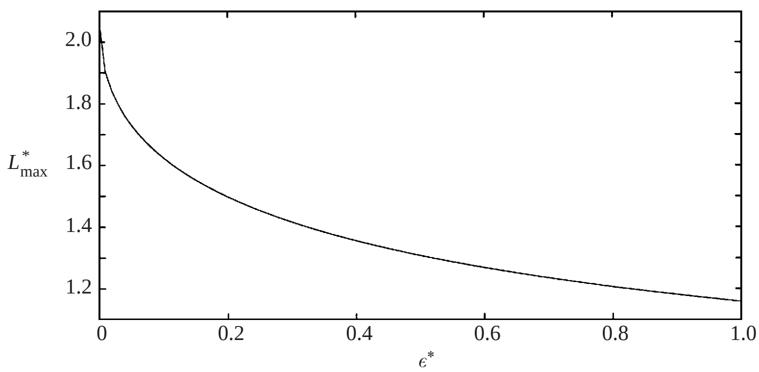

FIGURE 3. Maximum extent of current L^*_{max} as a function of ϵ^*.

regardless of the form of $h^*_1(t^*)$, we find the length of the current

$$L^*(t^*) = (1 - e^{-t^*})^{1/3}\sqrt{9^{2/3} + 6\epsilon^* e^{t^*}(1 - e^{-t^*})^{1/3} h^*_1(t^*)}, \tag{3.5}$$

and then, defining $\zeta = x^*/L^*(t^*)$, we may write $h^*(x^*, t^*) = h^*(0, t^*)(1 - \zeta^2)$.

In figure 2, we plot the length and volume of the draining current as a function of time for the cases $\epsilon^* = 0.1$ and $\epsilon^* = 0$. The main difference between the two results is the behaviour of length as a function of time: instead of asymptotically approaching a maximum run-out length, the current now retreats and drains out entirely in a finite time. The perturbation term corresponds to only a minor difference in the volume of the current; however, the length of the current is considerably reduced because drainage is always significant near its nose.

Figure 3 shows how the run-out length varies with ϵ^*. It is apparent that the component of drainage driven by the fluid weight significantly affects the flow even for small values of ϵ^*. It can be shown that for $\epsilon^* \ll 1$, the rescaled time τ^*_{max} at which the current reaches its maximum length is given by $\tau^*_{max} = 1 - 3^{1/3}\epsilon^{*1/2} + \cdots$, and thus $L_{max} = 9^{1/3} - 2\epsilon^{*1/2} + \cdots$ as $\epsilon^* \to 0$. Hence, the solutions obtained by Pritchard *et al.* (2001) somewhat overestimate the run-out length of the current: this is in accordance with the experimental results presented in that earlier study.

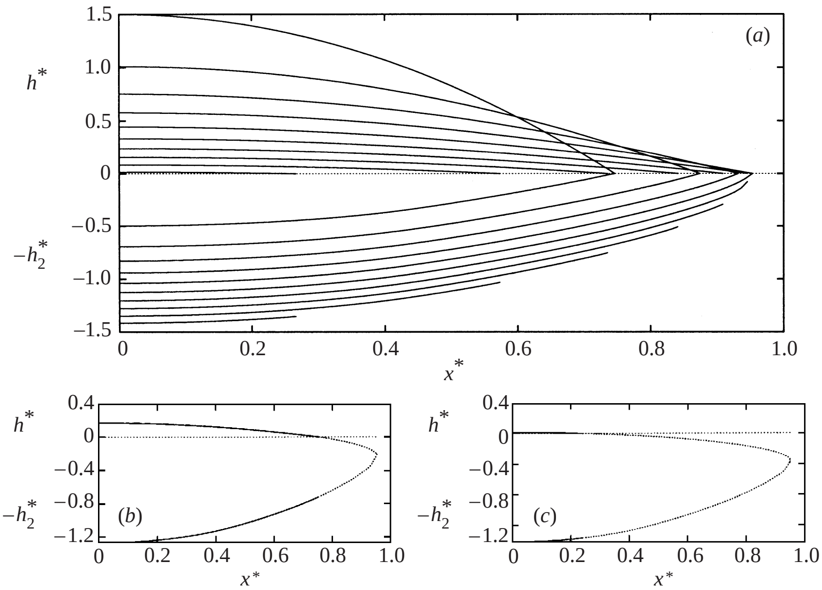

FIGURE 4. (a) Profiles of $h^*(x^*, t^*)$ and $-h_2^*(x^*, t^*)$ for $\lambda^* = 1$; plots at regular intervals from $t^* = 0.065$ to $t^* = 0.65$. The position of each interface at $x^* = 0$ descends as time increases. Lower plots show profiles of the current at (b) $t^* = 0.5$ and (c) $t^* = 0.64$, including both the region $0 \leqslant x^* \leqslant L^*(t^*)$ which the numerical method represents explicitly (—) and the region $x^* > L^*$ in which the lower layer drains under gravity (- -).

4. Drainage into a deep crack

We now consider the propagation of a current when the narrow drainage crack is deep on the scale of the current, and so never fully saturates. In order to investigate the instantaneous release of a finite volume of fluid, it is necessary to integrate the equations numerically, using an adaptive-grid method.

We can reduce the number of parameters in equations (2.19) by defining $\hat{h} = \lambda_1^{1/4} h^*$, $\hat{h}_2 = \lambda_1^{1/4} h_2^*$, $\hat{t} = \lambda_1^{-3/4} t^*$ and $\hat{x} = \lambda_1^{-1/4} x^*$. This yields the governing equations

$$\frac{\partial h^*}{\partial t^*} = \frac{\partial}{\partial x^*}\left(h^* \frac{\partial h^*}{\partial x^*}\right) - \frac{h^*}{h_2^*} - 1 \quad \text{and} \quad \frac{\partial h_2^*}{\partial t^*} = \lambda^* \frac{h^*}{h_2^*} + \lambda^*, \tag{4.1}$$

where $\lambda^* = \lambda_2/\lambda_1 = W/W_1$ (or in a porous system, $\lambda^* = \phi/\phi_1$). We first consider the behaviour of the system for $\lambda^* = 1$, and then examine the variation of the results with λ^*. (Recall that values of $\lambda^* \leqslant 1$ are physically meaningful only for the analagous system of flow through a porous medium, in which the porosity and permeability are independent quantities which depend on the microstructure of the porous matrix.)

Numerical integrations were carried out for various values of λ^* between 0.2 and 20. A consistent balance for $t^* \ll 1$ can be achieved if h^* tends to the similarity solution (2.21), so $h^* \sim t^{*-1/3}$, and $h_2^* \sim t^{*1/3}$; hence $h_2^* \to 0$ as $t^* \to 0$. However, it was more convenient to take an initial condition in which $h^*(x^*, t_0^*)$ was given by the similarity solution (2.21) at $t^* = t_0^*$, and to set $h_2^*(x^*, t_0^*) = h_{20}^* \approx 0.001$ to prevent numerical instability during the first few timesteps. The initial time t_0^* was typically 0.01, and the numerical results were not sensitive to the values of either t_0^* or h_{20}^*.

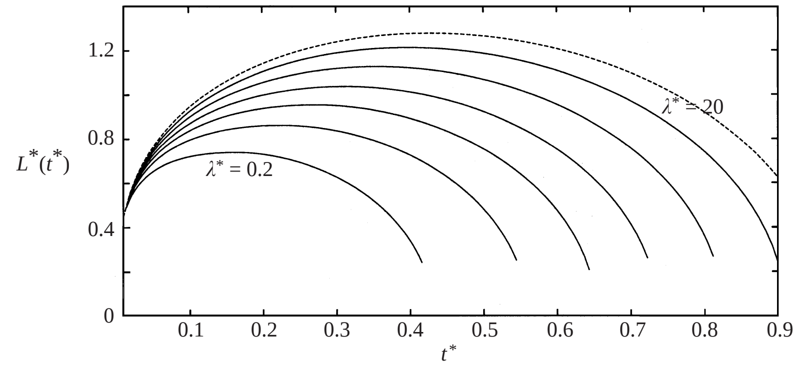

FIGURE 5. Length $L^*(t^*)$ of currents for $\lambda^* = 0.2$ to 20.0 (—) and analytical solution for $v^* = -1$ (- -).

4.1. *Results for* $\lambda^* = 1$

The evolution of the current for $\lambda^* = 1$ is illustrated in figure 4. The current in the upper layer behaves in a manner similar to the solution (3.2) for drainage through a thin layer: it advances to a maximum length and then retreats, draining out completely in a finite time. The profile of the upper current varies slightly in time, and is slightly more convex than the parabolic profile $h^* = h^*(0, t^*)(1 - \zeta^{*2})$.

In the lower layer, only the region $0 \leqslant x^* \leqslant L^*(t^*)$ is represented in our numerics: after the current has started to retreat, the fluid ahead of the nose of the current in the upper layer continues to drain through the lower layer at a steady rate $v^* = -1$, but no longer affects the dynamics in the upper layer.

The plots of the length and volume of the current are also qualitatively similar to those found for the current draining through a thin layer, and are omitted here for brevity. Drainage becomes increasingly dominant as the current evolves, and it is apparent that even by the time t^*_{max} at which the current reaches its maximum extent, the majority of the dense fluid lies in the crack and drains out of it at an almost constant rate mostly driven by its own weight.

4.2. *Variation of the results with* λ^*

We also investigate the behaviour of the current for $\lambda^* \neq 1$. Qualitatively it is very similar to that of the current with $\lambda^* = 1$, except that the rate at which the lower layer becomes deeper varies. The slight deviation from the parabolic profile is most marked for small λ^*, and least marked for large λ^*. Figure 5 compares $L^*(t^*)$ for currents with λ^* between 0.2 and 20.

As λ^* increases, the lower current becomes deeper more rapidly because of the decreased porosity of the lower layer. However, this does not correspond to a greater rate of mass loss from the upper current; in fact, since the pressure gradient across the lower current is smaller, the drainage rate from the upper current is reduced, and the current travels further and takes longer to drain out completely. As $\lambda^* \to \infty$, the ratio h^*/h^*_2 becomes negligible very rapidly; the drainage is almost entirely dominated by the constant term driven by fluid weight, and the behaviour of the current tends to the solution derived by Pritchard *et al.* (2001) for drainage driven by background flow, $v^* = -1$. In this limit, the run-out length L^*_{max} approaches $2^{3/4}3^{-1/4}$, and the drainage time t^*_{end} approaches $2^{3/4}3^{-1/2}$.

5. Summary and conclusions

Motivated by the problem of slurry injection into a fracture in a gas reservoir, we have developed a model for the flow of viscous fluid in a narrow vertical fracture, which then drains through a crack at its base. When the crack is much shallower than the current, it saturates immediately, and we are able to obtain an exact solution for the flow. When the crack is deeper, the evolution of the saturated region must be taken into account, although the overall behaviour of the current resembles that for drainage through a shallow crack. We have characterized the variation of the run-out length and drainage time with the relative widths of the crack and fracture.

An important point illustrated by these results is that the simplified drainage term employed by Pritchard *et al.* (2001) overpredicts the run-out length for a viscous fluid propagating either in a fracture or through a porous medium. This occurs because the drainage term used in the previous study is asymptotically valid only for small times, when the current is deep compared to the crack. However, the previous solutions provide an upper bound for the run-out length, as well as an estimate of the maximum time taken for the fluid to drain out altogether. These results have immediate applications in determining how far into a 'leaky' fracture a proppant slurry may be injected.

A natural development of the model developed here would be to consider the propagation of the current when the rheology is non-Newtonian (as is typical of most muds and slurries). A framework for these problems has recently been developed by King (2000), who obtained solutions for non-draining flows with power-law and Bingham rheologies.

The authors acknowledge financial support from EPSRC and HR Wallingford Ltd.

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